HW 3, PROBABILITY I

1. Let $\Omega = (0,1)$, $\mathcal{F}$ be Borel sigma algebra on Ω, and the probability be the standard Lebesgue measure. Show that the random variables $X_n(\omega) = sin(2\pi n\omega)$, $n = 1, 2, ...$ are not independent, but that for all $m, n \in \mathbb{N}$,

$$\mathbb{E}(X_n X_m) = \mathbb{E}X_n \mathbb{E}X_m.$$

2. Let $X_1, ..., X_n$ be i.i.d. standard normal random variables. Show that

$$\frac{X_1 + ... + X_n}{\sqrt{n}}$$

is a standard normal random variable.

3. Let X_1, X_2, X_3, X_4, X_5 be independent random variables, uniformly distributed on $[0,1]$. Find the distribution of $X_1 + X_2 + X_3 + X_4 + X_5$.

4. Find four random variables taking values in $\{-1, 1\}$ so that any three are independent but all four are not.

5. Let $X_1, X_2, ...$ be i.i.d. with $P(X_i > x) = \frac{e}{x \log x}$ for $x > e$. Show that $E|X_i| = \infty$, but there is a sequence of constants $\mu_n \to \infty$ so that $\frac{X_1 + ... + X_n}{n} - \mu_n$ converges to zero in probability.

6*. Let $f : [0,1] \to \mathbb{R}$ be a continuously differentiable function, and $\{f_n(x)\}_{n=1}^{\infty}$ be its Bernstein's polynomials. Show that $f_n'(x)$ tends uniformly to $f'(x)$, as $n \to \infty$.